AMERICAN
SOCIETY OF
CIVIL
ENGINEERS
FOUNDED
1852

STABILITY OF BEAM COLUMNS ABOVE
THE ELASTIC LIMIT

by Robert L. Ketter, J.M. ASCE

THIS PAPER

STABILITY OF BEAM COLUMNS ABOVE
THE ELASTIC LIMIT

Robert L. Ketter,[*] J.M., ASCE

SYNOPSIS

This paper, which discusses the behavior of pin-ended, steel, wide-flange beam columns, is the development of a virtual displacement method of solution to the inelastic instability problem. The method is equally applicable to the solution of elastic or inelastic beam-column problems but its greatest application seems to be in the inelastic field. Using the proposed method of solution, the critical value of loading for members subjected to concentric, eccentric and or lateral loads, can be determined in a fraction of the time required by existing solutions.

INTRODUCTION

The problem under consideration is the determination of the maximum loading a beam-column type member can carry. The importance of the problem is evident, since if it is possible to predict the "collapse" loading of a member, it is also possible to evaluate for a member with a given loading the true factor of safety.

A large number of variables enter in the solution of problems of this type, and to reduce these in the development that will follow, it will be assumed that:

    a. The mode of failure is that of bending in the plane of the applied moments. Furthermore, this plane is coincident with a principal axis of the section. (Lateral-torsional and/or local buckling are not considered.)

    b. The material under study possesses stress-strain properties as shown in Fig. 1.

    c. Members are originally straight, free from accidental eccentricities, and of uniform cross-section along their lengths.

To illustrate the phenomena being studied, consider the typical load-deflection curves of Figs. (2) and (3).

For the axially loaded member, Fig. 2 (elastic behavior), axial thrust, P, can be increased from zero to a certain critical value, P_e, with the member remaining in the straight equilibrium position. However, on application of the slightest additional increment of thrust above the critical value, lateral deformation occurs. Thus at the instant of bending there exist two adjacent positions of equilibrium of the same load; that is to say, the member is indifferent with regard to either a straight or a deflected configuration. This condition, known as bifurcation of the equilibrium position, is called a "condition of instability."

---

[*]Research Instructor, Fritz Eng. Lab., Lehigh Univ., Bethlehem, Pa.

For the eccentrically loaded column problem of Fig. 3 a different type of behavior is observed. First of all, the member starts to deflect on the slightest application of thrust. Elastically this deformation increases with the maximum strength approaching the Euler load. However, as deflections increase with load, the outermost compressive fiber stress at the center of the column increases as the product of P and y_0. Because of this rapidly increasing stress situation, there is reached a load at which yielding of the member takes place, (e.g., point "a" in Fig. 3). As the load is further increased, yielding progresses across and along the member, thereby reducing its resistance to further loading. Finally, there is reached a point at which an increase in load is impossible, the member progressively deflecting as load remains constant. At this instant, the member is indifferent with regard to two possible adjacent equilibrium configurations and it becomes unstable.

Herein then, "instability" is considered as that phenomenon where under a given constant load there exist two possible adjacent equilibrium positions.

To determine this condition of instability, several basic methods of solution are available. Three of these are as follows:

a. If the force required to displace the loaded column into a different, deflected position equals zero, then the member is said to be indifferent with regard to which of the positions it will assume, and a condition of instability exists.

b. If the total work associated with a loaded column remains the same as the member is deflected, that is, the rate of change with respect to deflection equals zero, indifference again has taken place.

c. If it is possible to determine the axial load - lateral deflection relation, the member will become unstable when

$$\frac{\partial P}{\partial y} = 0$$

since for that value of axial load there exist two adjacent equilibrium positions.

EXISTING SOLUTIONS

In the main, previous solutions of the inelastic instability, beam-column problem have been based on the criterion of instability that involves the determination of the maximum point of a load-deflection curve. In general each of those solutions has been based on the work of Karman, who, starting from an assumed strain distribution pattern, (and thus a stress distribution pattern), calculated the corresponding moment, thrust and curvature values which would have produced such a stress configuration. Since the three equations defining these variables, M, P and ϕ, contain five unknowns (see Fig. 4), one expression of the form

$$M_{int} = f(\phi, P) \tag{1}$$

can be obtained by eliminating the two cross-section variables, (α and σ_a), which define the generalized yield pattern.

Consider now a typical beam-column (Fig. 5). Since equilibrium must be satisfied at all times, i.e.

$$M_{ext} = M_{int} \tag{2}$$

and since the equation for external moment, M_{ext}, at any point along the column can be expressed as a function of P, y, lateral loads, etc., i.e.

$$M_{ext} = g\,(P,\ y,\ \text{lateral loads, etc.}),$$

solution to equation (1) will be of the form

$$P = h\,(\phi,\ y,\ \text{lateral loads, etc.}) \tag{3}$$

At this point the various methods of solutions differ. Karman and Chwalla[3] used numerical or graphical integration to solve equation (3) for direct values of P versus y. Jezek,[7] Westergaard and Osgood,[16] and others assumed a known deflection curve such that ϕ could be obtained directly in terms of y by differentiation. Both of these methods of solution have as their objective the determination of a means of describing thrust as a function of deflection,

$$P = f\,(y), \tag{4}$$

for which, according to criterion (c), instability occurs when

$$\frac{\partial P}{\partial y} = 0$$

(See Fig. 3.)

The difficulties involved in either of these methods of solution arise from the fact that for any conventional type of cross-section other than a simple rectangle or circle, it is impossible to write a continuous function for Eq. (1); therefore curves or graphs must first be devised to supply the M-P-ϕ information corresponding to various stress conditions.[9] Proceeding from these plots it is necessary to integrate suitable ϕ values and obtain the actual load-deflection curve.

To circumvent this difficulty, Jezek[7] introduced in his approximate solution a "shape factor"* the purpose of which was to relate the relative stiffness of the section in question to that of the rectangle. This type of solution lends itself readily to the formulation of design equations* which are comparatively easy to solve.

One notably different type of solution to the eccentric instability problem was presented by Ros,[11], [12] who reasoned that if the rate of increase in effective, internal, resistance eccentricity, M/P, is the same as the rate of increase in the external applied centerline eccentricity, then instability will occur. Using this condition he obtained a graphical solution for the rectangular section problem by expressing the internal eccentricity as a function of the sum of the outer fiber strains (a measure of curvature).

It has been shown that the presence of residual stress is of definite influence on the strength of columns.[6], [9] This makes solution by the above methods even more complicated.

To overcome the difficulties encountered in each of these solutions, the method presented in the next section of this paper was developed. The mechanics are similar to those presented by Ros[12] for the rectangular section.

---

*Page 45, Ref. (1).

THE PROPOSED VIRTUAL DISPLACEMENT
METHOD OF SOLUTION

Bending instability can be considered as a breakdown in the resistance the member offers to further bending. Since bending is synonymous with moment application, it should be possible, then, to determine for a given loading condition if the section offers sufficient resistance to further bending by displacing the loaded specimen a virtual amount and noting the change in external applied moment at the most critical section and the internal moment of resistance offered at that same section. If the increase in external moment is greater than that which the internal stress pattern can supply, the member fails since resisting moment cannot "keep up" with applied moments. When these two ratios of increase are the same a condition of indifference exists. The necessary condition for instability can then be stated:

When external moment at the most critical section increases at a rate equal to the rate of increase in resistance moment supplied by the internal stress system at that same section, the member becomes unstable.

Writing in the form of an equation

$$\left. \frac{\Delta M}{\Delta \phi} \right|_{ext} = \left. \frac{\Delta M}{\Delta \phi} \right|_{int} \tag{5}$$

where $\Delta M / \Delta \phi$ is the rate of change in moment with respect to curvature, ϕ.

For illustration consider the axially loaded column shown in Fig. 2 assumed, for ease of solution, to buckle in the elastic stress range. As the thrust, P, is increased the potential of the external system to subject the various cross-sections along the member to bending moment in addition to applied thrust increase as the thrust, i.e.

$$\Delta M_{ext} = P \Delta y \tag{6}$$

Whereas, internally (as shown in Fig. 6) the resistance to this possible external moment is constant at a value defined by the equation

$$\Delta M_{int} = EI \Delta \phi \tag{7}$$

ΔM_{ext} and ΔM_{int} represent changes in moment when going from the straight (original) to the deflected position (a virtual displacement away).

Assuming the deflection curve is of the form

$$\Delta y = \Delta y_0 \cos \frac{\pi x}{L} ,$$

then

$$\Delta \phi_0 = \frac{\partial^2}{\partial x^2} (\Delta y_0 \cos \frac{\pi x}{L}) = \Delta y_0 \frac{\pi^2}{L^2}$$

or

$$\Delta y_0 = \Delta \phi_0 \frac{L^2}{\pi^2} \tag{8}$$

where the zero subscript denotes conditions at the center of the column (the most critically deformed section). Substituting Eq. 8 in Eq. 6,

$$\Delta M_O \Big|_{ext} = P \frac{L^2}{\pi^2} \Delta \phi_O$$

Writing this expression and Eq. (7) as the change in moment divided by the virtual curvature to obtain the rate of change in moment with respect to ϕ_O gives

$$\frac{\Delta M_O}{\Delta \phi_O} \Big|_{ext} = P \frac{L^2}{\pi^2}$$

$$\frac{\Delta M_O}{\Delta \phi_O} \Big|_{int} = EI$$

(9)

Each of Eqs. 9 can be plotted as function of P and $(\Delta M/\Delta \phi)$, (see Fig. 7). There is reached a point at which the rate of external potentiality of moment application $(\Delta M/\Delta \phi |_{ext})$ exactly equals the rate of internal resistance to lateral deformation $(\Delta M/\Delta \phi |_{int})$. At that point the member can assume either the straight or the deflected position, and therefore a condition of instability exists. Above that critical value of load, P_{cr}, it is impossible for the internal change in resistance moment to "keep up" with the potential change in externally applied moments and therefore the member collapses. To evaluate the magnitude of the load corresponding to this critical condition, use the instability condition of Eq. 5,

$$\frac{\Delta M_O}{\Delta \phi_O} \Big|_{ext} = \frac{\Delta M_O}{\Delta \phi_O} \Big|_{int}$$

or

$$P_{cr} \frac{L^2}{\pi^2} = EI,$$

which when solved for P_{cr} gives

$$P_{cr} = \frac{\pi^2 EI}{L^2},$$

the Euler buckling load for a straight, axially loaded, elastic column.

Had the thrust been applied eccentrically and/or a lateral load superimposed thereby introducing bending from the start, the following would have resulted. In the original deflected equilibrium position (see Fig. 8), prior to subjecting the member to the virtual disturbance,

$$M_O \Big|_{ext} = P(e + y_O) + \frac{RL}{4}$$

whereas, after the Δy_O change

$$M_O' \Big|_{ext} = P(e + y_O + \Delta y_O) + \frac{RL}{4}$$

The increase in external moment caused by this disturbance is equal to the difference between the final and original moment values, i.e.

$$\Delta M_0 \bigg|_{ext} = M'_0 \bigg|_{ext} - M_0 \bigg|_{ext}$$

or

$$\Delta M_0 \bigg|_{ext} = P(\Delta y_0)$$

It should be noted that this is the same as for the axially loaded column problem shown previously.*

Internally, again considering elastic conditions for simplification, (see Fig. 9), resistance to bending is constant at a value

$$\frac{\Delta M}{\Delta \phi} \bigg|_{int} = EI$$

Solution to this problem then is the same as for the pure axial load case, instability occurring at the Euler buckling load. However from the discussion of Fig. 3 and from the bending resistance plot of Fig. 10 it is obvious that this condition can never be reached and satisfy the assumption of an elastic internal stress situation. To solve the problem then, it is first necessary to determine internal bending stiffness above the elastic limit.

Bending Stiffness Above the Elastic Limit

As was discussed in Section III (Fig. 4) the moment-curvature relation above the elastic limit is a function of axial thrust as well as cross-sectional dimensions and mechanical properties of the material. Thus to be able to use the resulting $M-\phi$ curves for integration it is necessary to first determine compatible moment and thrust quantities corresponding to given ϕ values. This can become a major problem for anything other than simple shapes; however, by modifying the problem under study to one where axial thrust is held constant and moments increased to collapse, a relatively simple, semigraphical solution is available[9] whereby compatible M, P, ϕ values can be determined for any type of cross-section having one axis of symmetry.

Two sets of curves resulting from that previous study are here shown as Fig. 11 and Fig. 12 and are the $M-\phi$ plots (at constant P-values) for the 8WF31 (strong axis bending) and for the rectangular section respectively. These plots have been made non-dimensional by dividing the moment value by M_y, the moment corresponding to initial yield in the absence of axial thrust, and the curvature value by ϕ_y, the curvature corresponding to M_y.

To have these results in a form suitable for direct use in Eq. 5 it is necessary to further develop them by obtaining tangent values at various points along each curve. These tangent values will be of the form

---

*Generalizing, it can be shown that if the member is pin-ended, regardless of lateral loads, inelastic or elastic action, etc.,

$$\Delta M \bigg|_{ext} = P(\Delta y) .$$

For the case of restrained members, a factor taking into account the end condition will need to be added to the right hand side of the equation.

$$\frac{\Delta M}{\Delta \phi} \cdot \frac{\phi_y}{M_y}$$

which can be considered as a measure of the desired internal stiffness term, $\Delta M/\Delta \phi |_{int}$, in Eq. 5. The tangent curves obtained from Figs. 11 and 12 are shown in Figs. 13 and 14. In each case the tangent values are plotted as a function of the two coordinates M/M_y and ϕ/ϕ_y.

If a cosine deflection curve is assumed and the external rate of change substituted in

$$\frac{\Delta M}{\Delta \phi} \cdot \frac{\phi_y}{M_y}\bigg|_{ext} = \frac{\Delta M}{\Delta \phi} \cdot \frac{\phi_y}{M_y}\bigg|_{int}$$

then,

$$\frac{\Delta M}{\Delta \phi} \frac{\phi_y}{M_y}\bigg|_{int} = \frac{P_{cr}}{P_e}, \tag{10}$$

since

$$\frac{\Delta M}{\Delta \phi} \cdot \frac{\phi_y}{M_y}\bigg|_{ext} = \frac{PL^2}{\pi^2} \cdot \frac{\phi_y}{M_y} = \frac{PL^2}{\pi^2} \cdot \frac{1}{EI} = \frac{P}{P_e}$$

In these equations, P_e is the Euler buckling load for the member of length L.

APPLICATIONS

For any member subjected to both end thrust and bending moments, the total moment at any section is composed of two parts: one is due to the axial thrust times the deflection; the other is independent of deflection (e.g. end moments, lateral loads, etc.). That part of the centerline moment which is independent of deflection is noted throughout the remainder of this discussion as "m_o."

If it is assumed that the moment producing loads are symmetrically placed on the structure and that the deflection curve can be approximated by the equation

$$y = y_o \cos \frac{\pi x}{L} \tag{11}$$

the external rate of increase in centerline moment with respect to curvature will be given by the equation

$$\frac{\Delta M}{\Delta \phi}\bigg|_{ext} = P \frac{L^2}{\pi^2}$$

However, to have this in the non-dimensional form suitable for use in Figs. 13 and 14, it is necessary to multiply both sides of the equation by $1/EI$. This then, as was shown in the last section, gives

$$\frac{\Delta M}{\Delta \phi} \cdot \frac{\phi_y}{M_y}\bigg|_{int} = \frac{P}{P_e} = \left(\frac{P}{P_y}\right)\left(\frac{\sigma_y}{\pi^2 E}\right)\left(\frac{L}{r}\right)^2 \tag{12}$$

where P_e is the Euler buckling load for the pin-ended member.

Since a cosine deflection curve (Eq. 11) was assumed, the centerline deflection can be expressed as

$$y_0 = (\frac{\phi_0}{\phi_y}) \frac{L^2}{\pi^2} \phi_y \tag{11a}$$

Substituting Eq. (11a) in the following expression for m_0

$$\frac{m_0}{M_y} = \frac{M_0}{M_y} - \frac{Py_0}{M_y}, \tag{13}$$

$$\frac{m_0}{M_y} = \frac{M_0}{M_y} - (\frac{P}{P_e})(\frac{\phi_0}{\phi_y}) \tag{14}$$

It should here be noted in the derivation of Eq. 14 it was not necessary to consider the explicit form of the moment producing loads; only the axial thrust, an assumed deflection curve and the length of the member was used. Eq. (14) then is the solution to the general inelastic instability problem for members, which due to applied loads, deform symmetrically according to Eq. (11).

To illustrate the procedure, consider the following numerical example: Given an 8WF31 section, $L/r = 80$, subjected to a constant axial thrust value of $P/P_y = 0.4$; determine the critical values of various "moment producing, loading components." (See Fig. 15.).

For this problem, according to Eq. (12), $\Delta M/\Delta\phi \cdot \phi_y/M_y|_{int} = (0.346)$, which from Figs. 13 and 14 give the critical centerline moment[*] and corresponding curvature[*] values as

$\frac{\Delta M}{\Delta\phi} \cdot \frac{\phi_y}{M_y}$		Strong Axis Bending	Weak Axis Bending (Rectangle)
0.346	$\frac{M_0}{M_y}$	0.624	0.904
	$\frac{\phi_0}{\phi_y}$	0.653	1.105

Substitution of these quantities in Eq. 14 gives the following values for m_0/M_y

---

[*]These values have been circled in Figs. 13 and 14.

	Strong Axis Bending	Weak Axis Bending (Rectangle)
$(\dfrac{m_o}{M_y})_{cr}$	0.398	0.522

Since in Fig. 15 the moment at the center of the column exclusive of thrust times deflection is equal to

$$m_o = (M_e + \frac{RL}{4}),$$

the critical value of the moment producing loads will be given by the following expressions:

$$(\frac{m_o}{M_y})_{cr} = \frac{(M_e + RL/4)}{M_y} \, cr = 0.398 \text{ (for strong axis bending)}$$
$$= 0.522 \text{ (for weak axis bending)}$$

Had other loading conditions been imposed, a similar procedure would have been used. The expressions for a few typical problems are shown in Table I.

In each of these problems, the quantity of primary importance is m_o/M_y. To facilitate solution of other problems where length and P/P_y values are different from this one, values of m_o/M_y have been obtained for both strong axis and weak axis bending in a range of slenderness from $0 \leq L/r \leq 120$. The plots resulting from these calculations are shown in Figs. 16 and 17. They are applicable only to those cases of load which produce single curvature deformation that can be approximated by a "full" cosine curve.

Figs. 16 and 17 can be used to solve three basically different types of beam-column problems.
1. Given a constant axial thrust, determine the maximum lateral load and/or end moment the member can carry.
2. Given the lateral loads and/or end moments, determine the maximum axial thrust to which the member can be subjected.
3. Given an eccentricity, determine the maximum axial thrust.
The first two are solvable directly from the figures. For the third case, however, it is first necessary to "lay-off" a line which described the relationship between m_o and P.* The point of intersection of this "e-line" with the given slenderness ratio will determine the critical axial thrust.

DISCUSSION AND SUMMARY

In the foregoing presentation a statical method was used for determining conditions of indifference. This method implies that an equilibrium position is known and it is desired to determine if that position is stable or unstable. Solution was achieved by comparing the rate of change in external moment to

---

*This line would correspond to the equation $m_o = P_e$ or

$$(m_o/M_y) = (P/P_y)(ec/r^2) .$$

the rate of change in internal resistance moment for a given virtual displacement of the loaded specimen. If the internal moment increased at a rate greater than the external moment, the specimen was stable. If the rate of change in external was the larger, the member was unstable. For the case where the rates of change were equal, a condition of indifference (incipient instability) existed.

a. Influence of Assumed Deflected Configuration

For ease of solution, it was assumed throughout this report that the overall behavior of the member could be predicted with sufficient accuracy by assuming a deflection curve, the magnitude of which would be governed by the strain distribution pattern at the most highly deformed cross-section. This simplification eliminated the need of minimizing integrals and resulted in a direct determination of an end moment, axial thrust and critical length corresponding to a given stress distribution pattern. However, when using such a procedure, a certain amount of error is introduced, the member essentially being forced into an unnatural configuration. To determine the magnitude of this error, the following three comparisons will be made.

    a. Comparison of strength results obtained for several different curves.

    b. Comparison of the deflection and strength results of each of these curves with individual solutions obtained by numerical integration, and

    c. Comparison with previous published solutions; namely, Chwalla's,[1] Jezek's[1] and Timoshenko's.[14]

In the first comparison, the three types of configuration shown in Fig. 18 are considered. The first of these, that of a "full-cosine" curve, was used in the development of the general method of solution (Section III) and also in the preceding section on "Applications." The second is a parabola, and the third is a "partial cosine" curve of the form suggested by Westergaard and Osgood.[16] In the first two cases the moment boundary condition is violated.

Since each of these assumed curves involves one parameter, y_0, the previously derived instability condition,

$$\left.\frac{\Delta M}{\Delta \phi}\right|_{ext} = \left.\frac{\Delta M}{\Delta \phi}\right|_{int} \quad ,$$

can be directly applied. For the first two cases,
    a. "Full Cosine" curve

$$\left.\frac{\Delta M}{\Delta \phi}\right|_{int} = P \frac{L^2}{\pi^2} \text{ , and} \tag{15}$$

    b. "Parabolic" curve

$$\left.\frac{\Delta M}{\Delta \phi}\right|_{int} = P \frac{L^2}{8} \tag{16}$$

For the "Partial Cosine" curve (assumption "c"), it is necessary to further develop $\Delta M/\Delta \phi\big|_{ext}$ so that the ends of the "L" length are held at a given deflection, e, rather than maintaining a hypothetical "ℓ" -length member. (See

Fig. 19). If such a procedure is carried out a complex equation containing functions of Δy_0 and $\Delta \ell$ and cosines thereof is obtained which cannot be solved explicitly for Δy_0 in terms of $\Delta \phi_0$. However, by considering only the first two terms of a cosine series, the expression can be reduced and results in the following equation:

$$\left. \frac{\Delta M}{\Delta \phi} \right|_{int} = P \frac{L^2}{\pi^2} \tag{17}$$

Even though this is the same external rate of moment increase as found for case "a," the solutions for members "a" and "c" are different, the M_e values being determined by different equations.

a. "Full Cosine" curve

$$M_e = M_0 - P \phi_0 \frac{L^2}{\pi^2} \tag{18}$$

b. "Partial Cosine" curve

$$M_e = M_0 \cos \frac{\pi}{2} \frac{L}{\ell} \tag{19}$$

ℓ is determined by considering the full value of M_0 and ϕ_0 at the center section, i.e.

$$\ell^2 = \frac{M_0}{\phi_0} \cdot \frac{\pi^2}{P} \tag{20}$$

$$\left(\frac{\ell}{r}\right)^2 = \frac{M_0/M_y}{(P/P_y)\,(\sigma y/\pi^2 E)} \tag{20a}$$

The comparison between these solutions is illustrated in Figs. 20 and 21. Here is presented a series of interaction curves for various length ratios for the 8WF31 member bent about both its strong and weak axes. The "full cosine" deflection curve gave the highest strength values. The "parabola" gave the lowest prediction.

A recent paper by John Clark[4] discusses, in part, this same question. Therein he not only investigates the applicability of the "partial cosine" curve but also a much more complicated three parameter deflection equation. Comparison with test results is also shown in that report. In general, however, his results indicate that the increase in accuracy obtained by using the more complicated deflection expression does not warrant the added work required to obtain a solution.

The second means of comparison, numerical integration, was used as a check in a previous report[9] and one figure based on that work is here included to indicate the observed trends. (See Fig. 22) There is difference between the load deflection curves obtained by the various methods; however, the ultimate carrying capacities of the members are in comparatively good agreement for these examples.

The third comparison is that between previous published solutions and the curves of ultimate strength as predicted by the "partial cosine" deflection curves presented earlier in this discussion. Since previous information regarding WF section behavior is lacking, comparison is made only for the

rectangular section; furthermore, only for two slenderness ratios.

Chwalla's solution is based on a numerical integration procedure and should therefore be considered as being "exact." His derivation is based on (a) a typical German steel with a stress-strain curve having a proportional limit of 27 ksi, and a yield stress level of 34 ksi. Jezek's is an approximation based on (a) a "full cosine" deflection curve and (b) a stress-strain curve of the idealized elastic type (see Fig. 1). Timoshenko's solution also considers (a) the "full cosine" curve but (b) the stress-strain curve has a proportional limit 30 ksi, a yield level of 36 ksi and an ultimate strength of 56 ksi.

Fig. 23 shows the comparison obtained between each of these previous solutions and the "partial cosine" solution of this report. All curves have been adjusted to a yield stress level of 40 ksi. Length adjustment was according to the equation

$$\left(\frac{L}{r}\right)_{adj} = \left(\frac{L}{r}\right)\sqrt{\frac{\sigma_y}{40.0}}$$

In general, Chwalla's solution gives the lowest strength values; Jezek's gives the highest. The partial cosine solution is between these two limits, as is Timoshenko's solution.

b. Elastically Restrained Members

In actual structures, columns are usually restrained at their ends rather than being pin-ended, and a recent paper[2] discusses such a condition. In that study a method was developed for reducing the actual restrained system to one of an equivalent, pin-ended, eccentrically loaded column. Having determined by that method the equivalent member for a restrained column problem, the solution and curves presented in this paper directly apply.

c. Residual Stresses

To include the variable of residual stress, it is necessary to develop a complementary set of curves for the desired pattern and magnitude of residual stress. This development would start from a given M-ϕ curve,[9] which included the effect of the residual stress pattern, and proceed along the same lines previously described.

d. Load-Deflection History

It should be pointed out that no special consideration has been given to the manner in which a condition of instability is approached; that is, the load-deflection history of the member in question. The presented curves and numerical work assume that an axial load, smaller than that required for failure, is first applied and that bending is then increased from zero to its maximum value. Under this condition there will be no unloading of previously yielded fibers. The resulting critical loads then will be comparable to the reduced modulus load for axially loaded members.

As was clearly demonstrated, first by Shanley[13] and later by Duberg and Wilder[5] and others, if an axially loaded column is considered as an imperfect compression member with the imperfection approaching zero, the column will start to bend at the tangent modulus load. Increased thrust beyond this value will be accompanied by an increasing rate of lateral deflection, the member reaching its maximum strength at some point prior to attainment of the reduced modulus load.

Since one limit of the generalized beam-column problem is the axially loaded column, there will be a certain amount of error involved in the method of solution developed in this report. While it is felt that for the assumed stress strain curve of Fig. 1, the resulting error is quite small, further extension of this work to materials having curved stress-strain relations (either as a basic material property or as a result of residual stress, etc.) should be accompanied by suitable check calculations to indicate the amount of error involved in this type of a procedure. The error will be primarily a function of the initial eccentricity. Other important factors would be the type of cross-section and the stress-strain relation.

A recent NACA report[17] provided one source of information for these check calculations. Therein load-deflection curves are presented (for an idealized WF section column) for various initial eccentricities and stress-strain curves.

Since a majority of the practical beam-column problems encountered in the analysis of Civil Engineering type structures contain moment producing loads greater than that which could be attributed to accidental imperfections, this "tangent modulus influence" will more than likely not be a real problem.

It is interesting to note that the early numerical work on collapse strength of rectangular section, beam-columns carried out by Chwalla was based on a typical "German Steel" type stress-strain relation (linear up to the proportional limit, parabolic from the proportional limit to the yield stress level and then horizontal to the point at which strain hardening commences). Recently[2] this work has been corrected and it was found that adjustment was needed only when the ratio of the centerline deflection at maximum load to the depth of the section was less than 0.1, a comparatively small deflection value.

e. Local and Lateral Instability

Neither local nor lateral types of instability were considered in the development presented in this paper. The considered condition of instability in each case has been that of excessive bending in the plane of the applied moment. However, when bending is forced about the major axis of a section (that condition most frequently encountered in practice); when there is appreciable difference in bending stiffness between the two principal axes of the section; and when there is little if any lateral support along the member, the mode of failure will more than likely be that due to a combined bending and twisting action.[8] In practice, however, this will not necessarily be the case since members will be frequently restrained by bracing, walls, floors, etc. Where sufficient lateral restraint is provided, the problem of bending plus twist is nonexistent, and collapse can be determined by the methods previously described.

Another entirely different type of problem can develop when the cross-section is composed of connected, relatively thin, projecting elements. For those types of members, the projecting elements may individually or collectively deform in certain local regions and thereby change the original cross-sectional properties of the member.

Assuming the validity of any of the currently available inelastic local buckling theories, the maximum strain, ϵ_{max}, that a flange can sustain and remain in the straight position can be predicted. From this can be determined the critical value of $\epsilon_{max}/\epsilon_y$. Using the procedure outlined in Ref. 9 and noting that

$$\frac{\epsilon_{max}}{\epsilon_y} = 1 + 2\alpha \left(\frac{\phi}{\phi_y}\right)$$

(see Fig. 24), the corresponding critical value of ϕ_0/ϕ_y for available, rolled, wide-flange shapes can be determined and is found to be comparatively large ($\phi_0/\phi_y \geq 3.0$). From Fig. 13 it can be seen that such a condition will be realized only at extremely low values of thrust, ($P_{cr}/P_e \leq 0.05$).

SUMMARY

This report presents a virtual displacement solution to the inelastic instability problem, the method being equally adaptable to elastic, inelastic, concentrically loaded, eccentrically loaded and or laterally loaded columns.

The essence of the solution is that of applying a virtual curvature change to the loaded member and equating the rate of change in external moment at the most highly stressed section to the internal stiffness resulting from that curvature change. For ease of solution, deflection curves of one parameter were assumed causing the system to reduce to collocation at one point. It was shown that the ultimate strength of such members is relatively unaffected by choice of deflection curve, the "full cosine" giving the highest strength prediction and the parabola giving the lowest prediction. (Figs. 20 and 21) The "partial cosine" solution compared favorably with previously available solutions. (Fig. 23)

To facilitate solution of problems other than those discussed herein, curves have been presented which ensure rapid determination of critical loading values (Figs. 20 and 21) for any symmetrically deformed pin-ended member.

The presented solutions of this paper apply only to mild structural steel and in the strickest sense only to the case where an axial thrust smaller than that required for failure is first applied and then held constant as bending moments are increased to collapse. This condition, however, can readily be approached in Civil Engineering Type Structures.

ACKNOWLEDGMENTS

A	Area of cross-section
b	Width of cross-section
d	Depth of cross-section
e	End eccentricity of axial load application
E	Young's modulus of elasticity
I	Moment of inertia of total cross-section about the principle axis of section under consideration
L	Length of member (distance between pins)
ℓ	Length of hypothetical member (used in "Partial Cosine" deflection curve assumption)
$\Delta\ell$	Change in length of hypothetical member due to a virtual change in centerline curvature
M	Bending moment
M_{int}	Internal bending moment due to stress distribution across section
M_{ext}	External bending moment due to lateral loads, end moment, initial curvature, and axial thrust times deflection
M_y	Bending moment at which yield point is reached in flexure
M_0	Bending moment at the centerline section of the member
m_0	That part of the centerline moment which is independent of deflection
M_e	End applied bending moment
ΔM_0	Change in value of centerline moment due to a virtual change in curvature at that same section
P	Axial thrust
P_y	Axial thrust corresponding to yield stress level over entire section
P_e	Euler buckling load
P_{cr}	Maximum thrust a member can carry
R	Concentrated lateral load
t	Flange thickness (wide-flange section)
u	Deflection parallel to web in undeformed position
v	Deflection perpendicular to web in undeformed position
y	Deflection in plane of moment application
y_0	Centerline deflection
Δy_0	Change in centerline deflection due to a virtual change in centerline curvature
α	Ratio of the depth of yield penetration in compression to total depth of section

ϵ	Unit strain
ϵ_y	Unit strain corresponding to initial yield
ϵ_{max}	Maximum average unit strain on compression flange of wide-flange section (strong axis bending)
σ	Unit normal stress
σ_y	Lower yield point stress
σ_a	Variable normal stress on tension side of cross-section (positive when in tension)
ϕ	Curvature
ϕ_0	Centerline curvature
$\Delta\phi_0$	Virtual change in centerline curvature

REFERENCES

1. Bleich, Friedrich, "Buckling Strength of Metal Structures," McGraw-Hill Book Company, New York, 1st Ed., (1952).

2. Bijlaard, P. P., Fisher, G. P., and Winter, G., "Strength of Columns Elastically Restrained and Eccentrically Loaded," PROCEEDINGS of the A.S.C.E., Separate 292, Oct. (1953).

3. Chwalla, E., "Der Einfluss der Querschnittsform auf das Tragvermogen aussermittig gedruckter Baustahlstabe," DER STAHBAU, Vol. 8, p. 193, (1935).

4. Clark, J. W., "Plastic Buckling of Eccentrically Loaded Aluminum Alloy Columns," PROCEEDINGS of the A.S.C.E., Separate 299, Oct. (1953).

5. Duberg, J. E., and Wilder, T. W., III, "Inelastic Column Theory," NACA TN2267, January (1951).

6. Huber, Alfons W., and Beedle, Lynn S., "Residual Stress and the Compressive Properties of Steel," Publication scheduled in THE WELDING JOURNAL.

7. Jezek, K., "Die Tragfahigkeit des gleichmabig querbelasteten Druckstabes aus ideal-plastischen Stahl," DER BAUTECHNIK, Vol. 5, p. 33, (1935).

8. Ketter, Robert L., Beedle, Lynn S., and Johnston, Bruce G., "Column Strength Under Combined Bending and Thrust," THE WELDING JOURNAL 31 (12), Research Supplement, 607-s to 622-s (1952).

9. Ketter, Robert L., Kaminsky, Edmund L., and Beedle, Lynn S., "Plastic Deformation of Wide-Flange Beam-Columns," PROCEEDINGS of the A.S.C.E., Separate 330, Oct. (1953).

10. Newmark, N. M., "Numerical Procedure for Computing Deflection Moments and Buckling Loads," TRANSACTIONS of the A.S.C.E., Vol. 108, p. 1161-1234, (1943).

11. Ros, M., "Die Knicksicherheit von an beiden Enden gelenkig gelagerten Stäben aus Konstruktionsstahl," PROOCEEDINGS of the 2nd. INTERNATIONAL CONGRESS OF APPLIED MECHANICS, Zurich, p. 368 (1926).

12. Ros, M., "Buckling of Columns," Translated and Distributed by the COLUMN RESEARCH COUNCIL, June (1951).

13. Shanley, F. R., "Inelastic Column Theory," JOURNAL OF AERONAUTICAL SCIENCE, Vol. 14, No. 5, pp. 261-267, May (1947).

14. Timoshenko, S., "Theory of Elastic Stability" McGraw-Hill Book Company, New York (1936).

15. Wastlund, George, and Bergstrom, Sven G., "Buckling of Compressed Steel Members," TRANSACTIONS of the ROYAL INSTITUTE OF TECHNOLOGY, Stockholm, Sweden (1949).

16. Westergaard, H. M., and Osgood, W. R., "Strength of Columns," TRANSACTIONS of the A.S.M.E., Vols. 49, 50, APM-50-9, p. 65 (1928).

17. Wilder, T. W., III, Brooks, W. A., Jr., and Mathauser, E. E., "The Effect of Initial Curvature on the Strength of an Inelastic Column," NACA TN2872, January (1953).
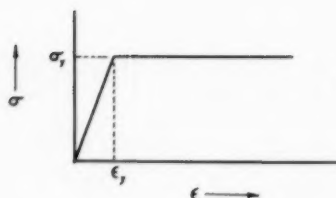
FIG. 1 IDEALIZED STRESS-STRAIN RELATION

FIG. 2 TYPICAL LOAD-DEFLECTION RELATION FOR AXIALLY LOADED COLUMNS

Typical Elastic Behavior

Typical Inelastic Behavior

$$\left[\frac{L^2}{r^2} < \frac{\pi^2 E}{\sigma_y}\right]$$

Elastic Behavior

"a"

Typical Inelastic Behavior
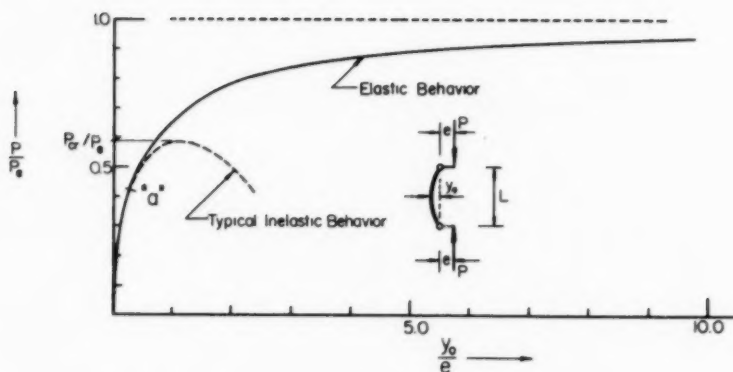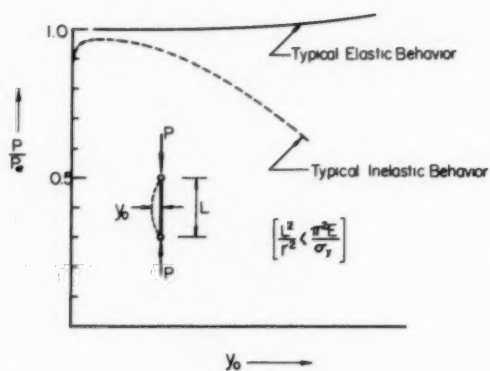
FIG. 3 LOAD-DEFLECTION RELATION FOR ECCENTRICALLY LOADED COLUMNS [9]

Cross-Section   Stress Distribution

$$M = \int_A \sigma\, y\, dA$$
$$P = \int_A \sigma\, dA$$
$$E\phi = \frac{d\sigma}{dy}\Big|_{neutral\ axis}$$
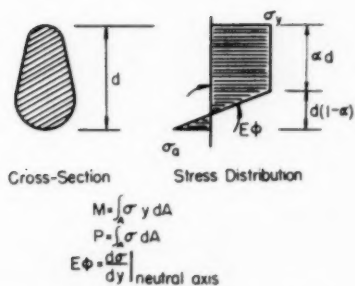
FIG. 4 BASIC INTERNAL STRESS-
LOAD RELATIONS
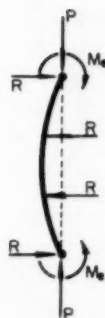
FIG. 5 TYPICAL BEAM-COLUMN

$$\Delta M_{int} = \int_A \sigma\, y\, dA = EI\Delta\phi$$
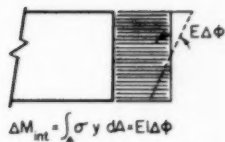
FIG. 6 CHANGE IN STRESS
DISTRIBUTION AND RESISTING
MOMENT DUE TO A VIRTUAL
CHANGE IN CURVATURE
(ELASTIC-AXIAL LOAD)

A Measure of the
Resistance to Bending

$$\frac{\Delta M}{\Delta\phi}\Big|_{int} = P\frac{L^2}{\pi^2}$$

$$\frac{\Delta M}{\Delta\phi}\Big|_{int} = EI$$
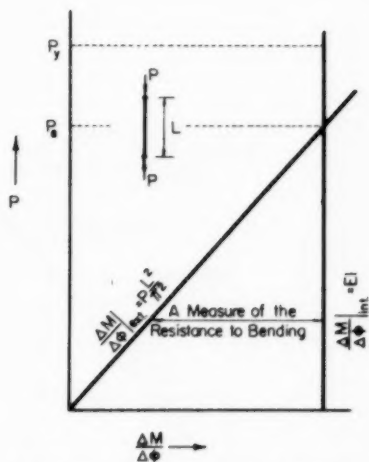
FIG. 7 TYPICAL BENDING RESISTANCE
RELATION FOR AXIALLY LOADED,
ELASTIC COLUMNS
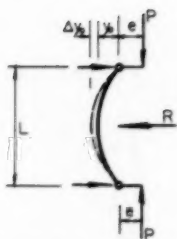
FIG. 8 TYPICAL BEAM-COLUMN
SUBJECTED TO VIRTUAL
DISPLACEMENT
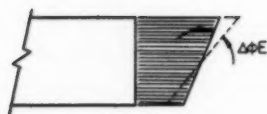
$$\Delta M_{rst} = \int_A \sigma\, y\, dA = EI\, \Delta\Phi$$

FIG. 9 CHANGE IN STRESS DISTRIBUTION
AND RESISTING MOMENT DUE TO A
VIRTUAL CHANGE IN CURVATURE
(ELASTIC-ECCENTRIC LOAD)
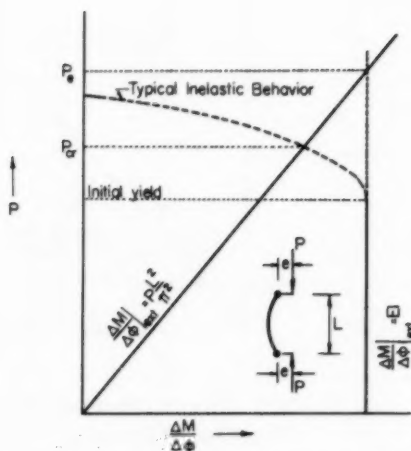
FIG. 10 TYPICAL BENDING RESISTANCE RELATION FOR
ECCENTRICALLY LOADED COLUMNS
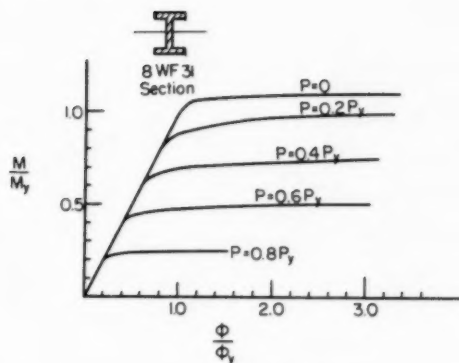
FIG. II MOMENT-CURVATURE RELATIONS FOR THE
8 WF 3I SECTION, STRONG AXIS BENDING [9]
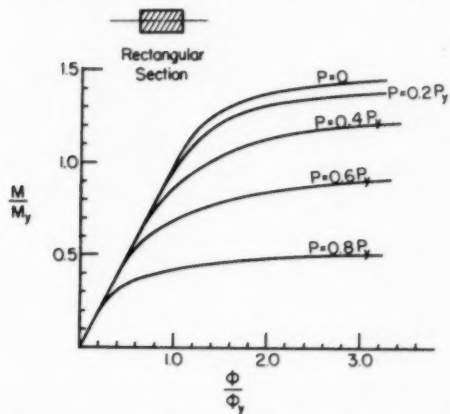
FIG. 12 MOMENT-CURVATURE RELATIONS FOR
THE RECTANGULAR SECTION [9]
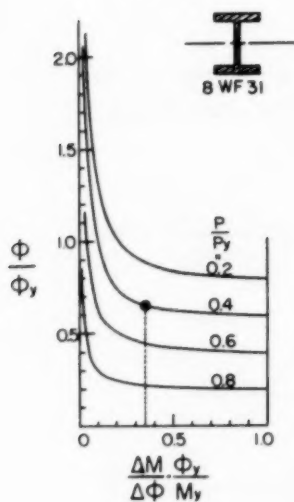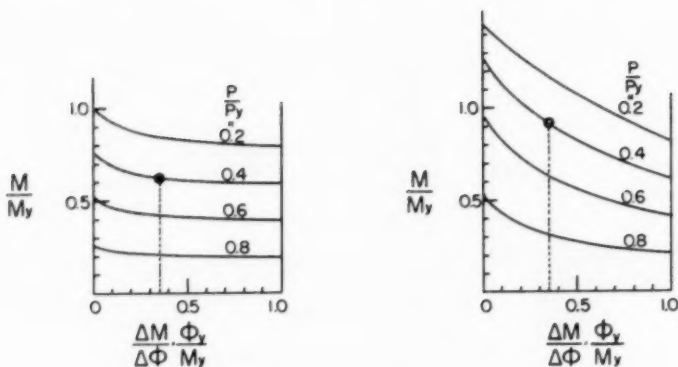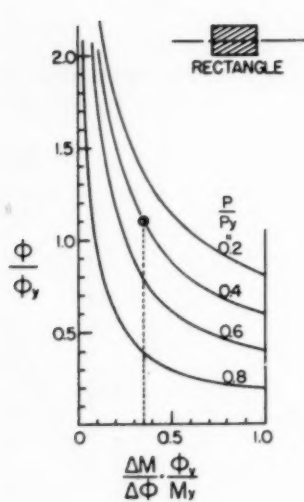
FIG. 13   FIG. 14

INTERNAL RATE OF CHANGE OF MOMENT VERSUS
MOMENT AND CURVATURE FOR CONSTANT AXIAL
THRUST VALUES
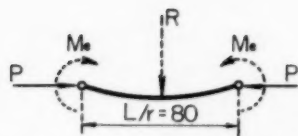
FIG. 15 TYPICAL BEAM-COLUMN
(used in example problem)
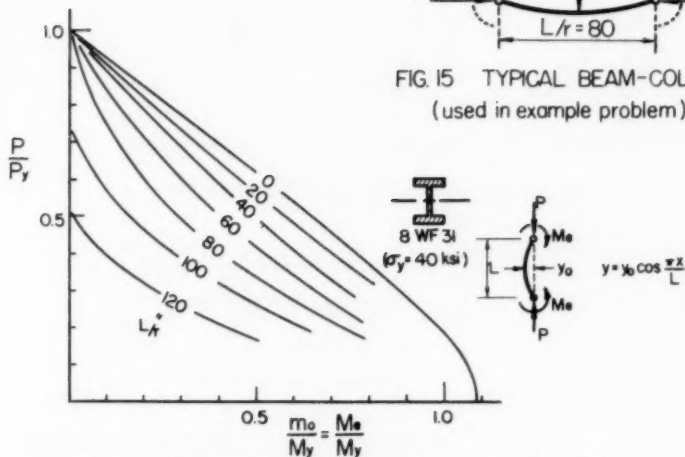
FIG. 16 INTERACTION CURVES FOR 8 WF 31 SECTION
(STRONG AXIS BENDING)
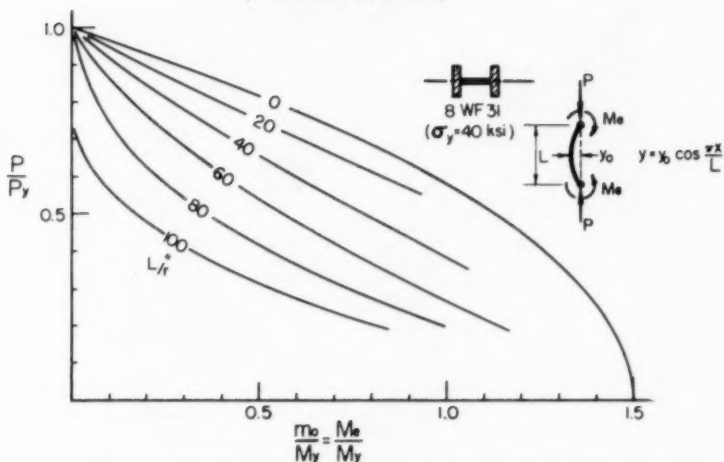
FIG. 17 INTERACTION CURVES FOR 8 WF 31 SECTION
(WEAK AXIS BENDING)

FIG.18 ASSUMED DEFLECTION CURVES

(a) $y = y_0 \cos \pi x / L$

(b) $y = y_0 [1 - 4x^2/L^2]$

(c) $y = y_0 \cos \pi x / l$
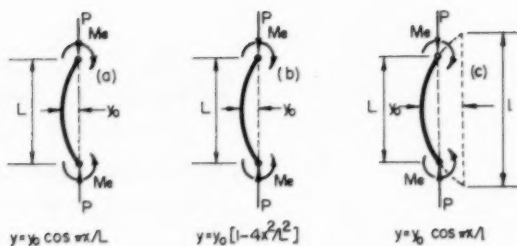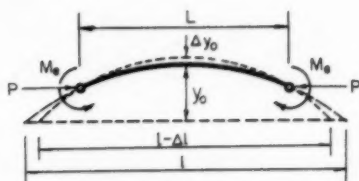
FIG. 19 VIRTUAL DEFLECTION FOR "PARTIAL COSINE" ASSUMPTION

------- ASSUMED "FULL COSINE" DEFLECTION CURVE
——— ASSUMED "PARTIAL COSINE" DEFLECTION CURVE
—·—·— ASSUMED "PARABOLIC" DEFLECTION CURVE

8 WF 31
(σ_y = 40 ksi)

$$\frac{m_0}{M_y} = \frac{M_e}{M_y}$$
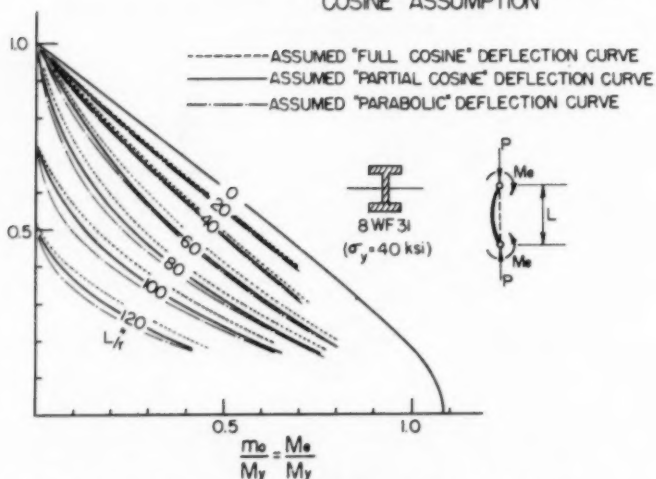
FIG. 20 INTERACTION CURVES SHOWING INFLUENCE OF
DEFLECTION CURVE ASSUMPTION
(STRONG AXIS BENDING)

692-24

FIG. 21 INTERACTION CURVES SHOWING INFLUENCE OF DEFLECTION CURVE ASSUMPTION (WEAK AXIS)

FIG. 22 LOAD-DEFLECTION CURVES SHOWING INFLUENCE OF DEFLECTION CURVE ASSUMPTION (WEAK AXIS BENDING)
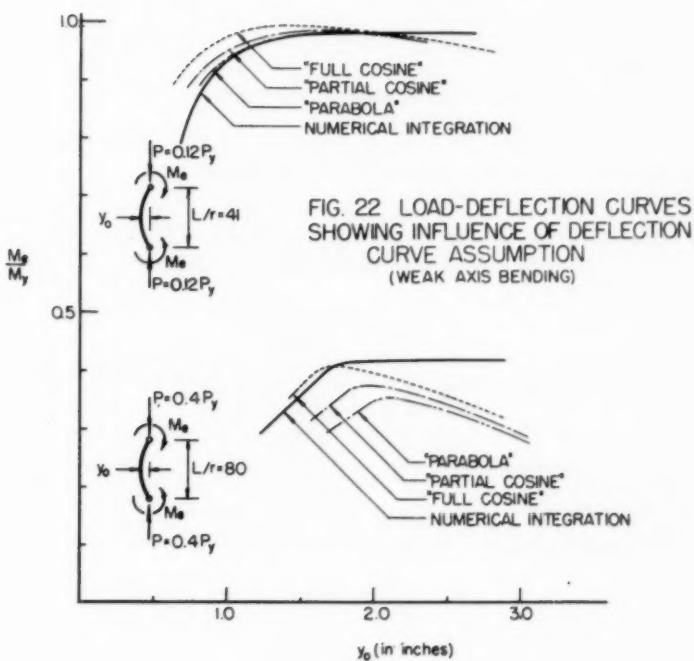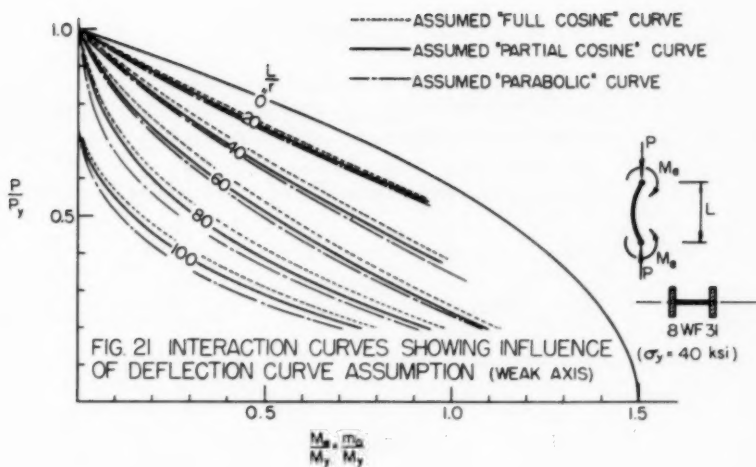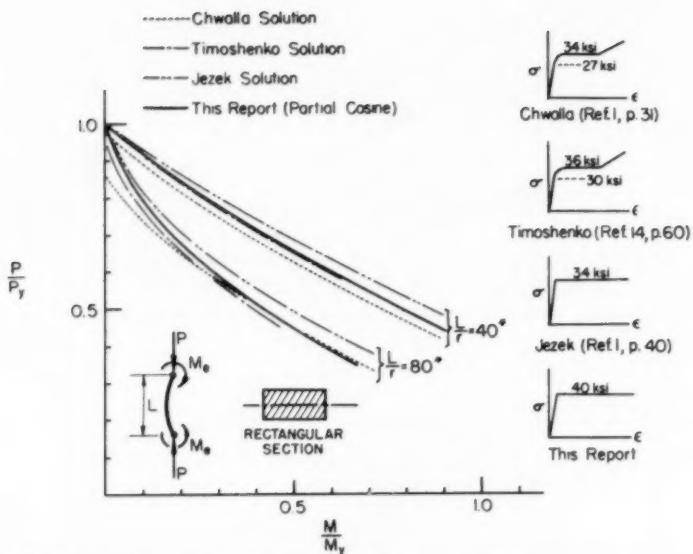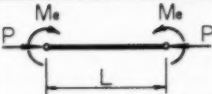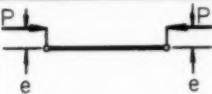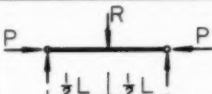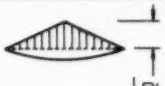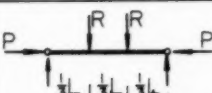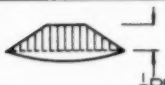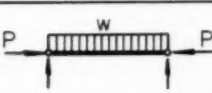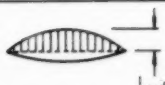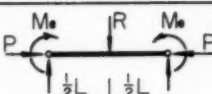
* Lengths have been adjusted to account for differences in yield level.
Curves correspond to σ_y=40 ksi

FIG. 23 INTERACTION CURVES SHOWING COMPARISON
WITH PREVIOUS SOLUTIONS

FIG. 24 STRESS DISTRIBUTION PATTERN WHEN
STRAINS EXCEED THE ELASTIC LIMIT
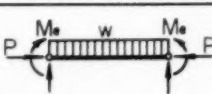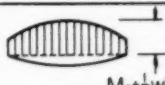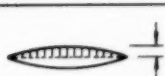
TABLE I

LOADING CONDITION	MOMENT DIAGRAM	DETERMINATION OF UNKNOWN
P — M_e ... M_e — P, span L	M_e	$\left(\dfrac{m_o}{M_y}\right)_{cr} = \left(\dfrac{M_e}{M_y}\right)_{cr}$
P, eccentric load e ... P, e	Pe	$\left(\dfrac{m_o}{M_y}\right)_{cr} = \dfrac{P(e)_{cr}}{M_y}$ *
P — R at center — P, $\frac{1}{2}L$, $\frac{1}{2}L$	$\frac{1}{4}RL$	$\left(\dfrac{m_o}{M_y}\right)_{cr} = R_{cr}\left[\dfrac{L}{4M_y}\right]$
P — R, R — P, $\frac{1}{3}L$, $\frac{1}{3}L$, $\frac{1}{3}L$	$\frac{1}{3}RL$	$\left(\dfrac{m_o}{M_y}\right)_{cr} = R_{cr}\left[\dfrac{L}{3M_y}\right]$
P — w (distributed) — P	$\frac{1}{8}wL^2$	$\left(\dfrac{m_o}{M_y}\right)_{cr} = w_{cr}\left[\dfrac{L^2}{8M_y}\right]$
P — M_e, R, M_e — P, $\frac{1}{2}L$, $\frac{1}{2}L$	$M_e+\frac{1}{4}RL$	$\left(\dfrac{m_o}{M_y}\right)_{cr} = \dfrac{(M_e+\frac{1}{4}RL)_{cr}}{M_y}$
P — M_e, w, M_e — P	$M_e+\frac{1}{8}wL^2$	$\left(\dfrac{m_o}{M_y}\right)_{cr} = \dfrac{(M_e+\frac{1}{8}wL^2)_{cr}}{M_y}$
P — a_o — P	Pa_o	$\left(\dfrac{m_o}{M_y}\right)_{cr} = \dfrac{Pa_o}{M_y} = \left[\left(\dfrac{P}{P_y}\right)\dfrac{c}{r^2}(a_o)_{cr}\right]$

* $\left(\dfrac{ec}{r^2}\right) = \left[\dfrac{m_o/M_y}{P/P_y}\right]$

692-27

The technical papers published in the past year are presented below. Technical-division sponsorship is indicated by an abbreviation at the end of each Separate Number, the symbols referring to: Air Transport (AT), City Planning (CP), Construction (CO), Engineering Mechanics (EM), Highway (HW), Hydraulics (HY), Irrigation and Drainage (IR), Power (PO), Sanitary Engineering (SA), Soil Mechanics and Foundations (SM), Structural (ST), Surveying and Mapping (SU), and Waterways (WW) divisions. For titles and order coupons, refer to the appropriate issue of "Civil Engineering" or write for a cumulative price list.

VOLUME 80 (1954)

MAY: 435(SM), 436(CP)^C, 437(HY)^C, 438(HY), 439(HY), 440(ST), 441(ST), 442(SA), 443(SA).

JUNE: 444(SM)^e, 445(SM)^e, 446(ST)^e, 447(ST)^e, 448(ST)^e, 449(ST)^e, 450(ST)^e, 451(ST)^e, 452(SA)^e, 453(SA)^e, 454(SA)^e, 455(SA)^e, 456(SM)^e.

JULY: 457(AT), 458(AT), 459(AT)^C, 460(IR), 461(IR), 462(IR), 463(IR)^C, 464(PO), 465(PO)^C.

AUGUST: 466(HY), 467(HY), 468(ST), 469(ST), 470(ST), 471(SA), 472(SA), 473(SA), 474(SA), 475(SM), 476(SM), 477(SM), 478(SM)^C, 479(HY)^C, 480(ST)^C, 481(SA)^C, 482(HY), 483(HY).

SEPTEMBER: 484(ST), 485(ST), 486(ST), 487(CP)^C, 488(ST)^C, 489(HY), 490(HY), 491(HY)^C, 492(SA), 493(SA), 494(SA), 495(SA), 496(SA), 497(SA), 498(SA), 499(HW), 500(HW), 501(HW)^C, 502(WW), 503(WW), 504(WW)^C, 505(CO), 506(CO)^C, 507(CP), 508(CP), 509(CP), 510(CP), 511(CP).

OCTOBER: 512(SM), 513(SM), 514(SM), 515(SM), 516(SM), 517(PO), 518(SM)^C, 519(IR), 520(IR), 521(IR), 522(IR)^C, 523(AT)^C, 524(SU), 525(SU)^C, 526(EM), 527(EM), 528(EM), 529(EM), 530(EM)^C, 531(EM), 532(EM)^C, 533(PO).

NOVEMBER: 534(HY), 535(HY), 536(HY), 537(HY), 538(HY)^C, 539(ST), 540(ST), 541(ST), 542(ST), 543(ST), 544(ST), 545(SA), 546(SA), 547(SA), 548(SM), 549(SM), 550(SM), 551(SM), 552(SA), 553(SM)^C, 554(SA), 555(SA), 556(SA), 557(SA).

DECEMBER: 558(ST), 559(ST), 560(ST), 561(ST), 562(ST), 563(ST)^C, 564(HY), 565(HY), 566(HY), 567(HY), 568(HY)^C, 569(SM), 570(SM), 571(SM), 572(SM)^C, 573(SM)^C, 574(SU), 575(SU), 576(SU), 577(SU), 578(HY), 579(ST), 580(SU), 581(SU), 582(Index).

VOLUME 81 (1955)

JANUARY: 583(ST), 584(ST), 585(ST), 586(ST), 587(ST), 588(ST), 589(ST)^C, 590(SA), 591(SA), 592(SA), 593(SA), 594(SA), 595(SA)^C, 596(HW), 597(HW), 598(HW)^C, 599(CP), 600(CP), 601(CP), 602(CP), 603(CP), 604(EM), 605(EM), 606(EM)^C, 607(EM).

FEBRUARY: 608(WW), 609(WW), 610(WW), 611(WW), 612(WW), 613(WW), 614(WW), 615(WW), 616(WW), 617(IR), 618(IR), 619(IR), 620(IR), 621(IR)^C, 622(IR), 623(IR), 624(HY)^C, 625(HY), 626(HY), 627(HY), 628(HY), 629(HY), 630(HY), 631(HY), 632(CO), 633(CO).

MARCH: 634(PO), 635(PO), 636(PO), 637(PO), 638(PO), 639(PO), 640(PO), 641(PO)^C, 642(SA), 643(SA), 644(SA), 645(SA), 646(SA), 647(SA)^C, 648(ST), 649(ST), 650(ST), 651(ST), 652(ST), 653(ST), 654(ST)^C, 655(SA), 656(SM)^C, 657(SM)^C, 658(SM)^C.

APRIL: 659(ST), 660(ST), 661(ST)^C, 662(ST), 663(ST), 664(ST)^C, 665(HY)^C, 666(HY), 667(HY), 668(HY), 669(HY), 670(EM), 671(EM), 672(EM), 673(EM), 674(EM), 675(EM), 676(EM), 677(EM), 678(HY).

MAY: 679(ST), 680(ST), 681(ST), 682(ST)^C, 683(ST), 684(ST), 685(SA), 686(SA), 687(SA), 688(SA), 689(SA)^C, 690(EM), 691(EM), 692(EM), 693(EM), 694(EM), 695(EM), 696(PO), 697(PO), 698(SA), 699(PO)^C, 700(PO), 701(ST)^C.

c. Discussion of several papers, grouped by Divisions.
e. Presented at the Atlantic City (N.J.) Convention in June, 1954.

AMERICAN SOCIETY OF CIVIL ENGINEERS

OFFICERS FOR 1955

PRESIDENT
WILLIAM ROY GLIDDEN

VICE-PRESIDENTS

Term expires October, 1955:
ENOCH R. NEEDLES
MASON G. LOCKWOOD

Term expires October, 1956:
FRANK L. WEAVER
LOUIS R. HOWSON

DIRECTORS

Term expires October, 1955:
CHARLES B. MOLINEAUX
MERCEL J. SHELTON
A. A. K. BOOTH
CARL G. PAULSEN
LLOYD D. KNAPP
GLENN W. HOLCOMB
FRANCIS M. DAWSON

Term expires October, 1956:
WILLIAM S. LaLONDE, JR.
OLIVER W. HARTWELL
THOMAS C. SHEDD
SAMUEL B. MORRIS
ERNEST W. CARLTON
RAYMOND F. DAWSON

Term expires October, 1957:
JEWELL M. GARRELTS
FREDERICK H. PAULSON
GEORGE S. RICHARDSON
DON M. CORBETT
GRAHAM P. WILLOUGHBY
LAWRENCE A. ELSENER

PAST-PRESIDENTS
Members of the Board

WALTER L. HUBER

DANIEL V. TERRELL

EXECUTIVE SECRETARY
WILLIAM H. WISELY

TREASURER
CHARLES E. TROUT

ASSISTANT SECRETARY
E. L. CHANDLER

ASSISTANT TREASURER
CARLTON S. PROCTOR

PROCEEDINGS OF THE SOCIETY

HAROLD T. LARSEN
Manager of Technical Publications

DEFOREST A. MATTESON, JR.
Editor of Technical Publications

PAUL A. PARISI
Assoc. Editor of Technical Publications

COMMITTEE ON PUBLICATIONS

SAMUEL B. MORRIS, *Chairman*

JEWELL M. GARRELTS, *Vice-Chairman*

GLENN W. HOLCOMB

OLIVER W. HARTWELL

ERNEST W. CARLTON

DON M. CORBETT